BGD COLLEGE ,KESAIBAHAL

Blended learning modules

2nd Year 3rd SEM

Subject and paper :- **MATHEMATICAL PHYSICS-II (PAPER-V)**

Series Solⁿ of D.E

The solⁿ of a D.E can be expressed as the sum of infinite term of a Series.

Ordinary Point

consider an Eqⁿ

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = 0$$

if for $x=a$, P and Q are well defined and their denominator do not vanish.

Then $x=a$ called an ordinary Point.

Ex $(1+x^2)y'' + xy' - y = 0$

$$y'' + \frac{x}{1+x^2}y' - \frac{y}{1+x^2} = 0$$

$$P = \frac{x}{1+x^2}, \quad Q = \frac{-1}{1+y^2}$$

at $x=0$

$P, Q \neq \infty$

Hence $x=0$ is an ordinary Point.

Solve the series solution of $\frac{d^2y}{dx^2} + x^2y = 0$

we have

$$y'' + x^2y = 0$$

$$P = 0, \quad Q = x^2$$

As P, Q are well defined at $x=0$, then $x=0$ is an ordinary Point.

Let the solution to be

$$y = \sum_{m=0}^{\infty} A_m x^m$$

$$y' = \sum_{m=0}^{\infty} m A_m x^{m-1}$$

$$y'' = \sum_{m=0}^{\infty} m(m-1) A_m x^{m-2}$$

on substituting in eq^n ① we get

$$m(m-1) \sum_{m=0}^{\infty} A_m x^{m-2} + \sum_{m=0}^{\infty} A_m x^{m+2} = 0$$

Now equating to zero the coefficient of least power of x.

i) i.e coff^n of x^0

$$2(2-1) A_2 = 0$$

$$\boxed{A_2 = 0}$$

ii) i.e coff^n of x^1

$$3(3-1) A_3 = 0$$

$$\boxed{A_3 = 0}$$

iii) i.e coff^n of x^2

$$4(4-1) A_4 + A_0 = 0$$

$$\boxed{A_4 = -\frac{1}{12} A_0}$$

Now Equating to zero the coefficient of x^r, we get

$$(r+2)(r+1) A_{r+2} + A_{r-2} = 0$$

$$A_{r+2} = - \frac{A_{r-2}}{(r+1)(r+2)}$$

$$A_4 = - \frac{A_0}{4 \cdot 3} = - \frac{1}{12} A_0$$

$$A_5 = - \frac{A_1}{4 \cdot 5}$$

$$A_6 = - \frac{A_2}{5 \cdot 6} = 0$$

$$A_7 = - \frac{A_3}{6 \cdot 7} = 0$$

$$A_8 = - \frac{A_4}{7 \cdot 8} = \frac{A_0}{3 \cdot 4 \cdot 7 \cdot 8}$$

$$A_9 = - \frac{A_5}{8 \cdot 9} = \frac{A_1}{4 \cdot 5 \cdot 8 \cdot 9}$$

Hence the series solⁿ is

$$y = \sum_{m=0}^{\infty} A_m x^m$$

$$y = A_0 + A_1 x - \frac{1}{3 \cdot 4} A_0 x^4 - \frac{A_1}{4 \cdot 5} x^5 + \frac{A_0}{3 \cdot 4 \cdot 7 \cdot 8} x^8 + \frac{A_1}{4 \cdot 5 \cdot 8 \cdot 9} x^9 \ldots$$

$$y = A_0 \left(1 - \frac{x^4}{3 \cdot 4} + \frac{x^8}{3 \cdot 4 \cdot 7 \cdot 8} + \cdots \right) + A_1 \left(x - \frac{x^5}{4 \cdot 5} + \frac{x^9}{4 \cdot 5 \cdot 8 \cdot 9} \cdots \right)$$

Ans.

Find the Power Series Solution of $(1-x^2)y'' - 2xy' + 2y = 0$ about $x = 0$.

We have

$$(1-x^2)y'' - 2xy' + 2y = 0 \quad \text{——— (1)}$$

$$y'' - \frac{2x}{(1-x^2)}y' + \frac{2y}{(1-x^2)} = 0$$

P, Q are well defined at $x=0$, hence $x=0$ is an ordinary point.

Let the series sol^n be

$$y = \sum_{m=0}^{\infty} a_m x^m$$

$$y' = m \sum_{m=0}^{\infty} a_m x^{m-1}$$

$$y'' = m(m-1) \sum_{m=0}^{\infty} a_m x^{m-2}$$

from eq^n (1), we get

$$m(m-1)\sum_{m=0}^{\infty} a_m x^{m-2} - m(m-1)\sum_{m=0}^{\infty} a_m x^m - 2m\sum_{m=0}^{\infty} a_m x^m$$

$$+ 2\sum_{m=0}^{\infty} a_m x^m = 0$$

$$m(m-1)\sum_{m=0}^{\infty} a_m x^{m-2} + [2 - 2m - m(m-1)]\sum_{m=0}^{\infty} a_m x^m = 0$$

Now Eq^n to zero the coeff^n of x^0

$$2(2-1)a_2 + [2]a_0 = 0$$

$$2a_2 + 2a_0 = 0$$

$$\boxed{a_2 = -a_0}$$

coff^n of x^1

$$3\cdot 2\, a_3 + (2-2-0]\, a_1 = 0$$

$$\boxed{a_3 = 0}$$

Now Eq^n to zero the coeff^n of x^r

$$(r+2)(r+1)\,a_{r+2} + [2-2r-r(r-1)]\,a_r = 0$$

$$a_{r+2} = + \frac{a_r\,[2(r-1)+r(r-1)]}{(r+2)(r+1)}$$

$$\boxed{a_{r+2} = \frac{r-1}{r+1}\,a_r}$$

(i) $r=0$

$$\boxed{a_2 = -a_0}$$

(ii) $r=1$

$$\boxed{a_3 = 0}$$

(iii) $r=2$

$$a_4 = \frac{1}{3}a_2 = -\frac{a_0}{3}$$

$$\boxed{a_4 = -\frac{a_0}{3}}$$

(iv) $r=3$

$$a_5 = \frac{2}{4}a_3 = \frac{1}{2}a_3 = 0$$

$$\boxed{a_5 = 0}$$

$$a_7 = a_9 = a_{11} = \cdots = 0$$

(v) $r=4$

$$a_6 = \frac{3}{5}a_4$$

$$\boxed{a_6 = -\frac{1}{5}a_0}$$

Hence the series soln is given by.

$$y = a_0\left[1 - x^2 - \frac{1}{3}x^4 - \frac{1}{5}x^6 + \cdots\right] + a_1 x$$

solved

Singular Points

consider the Equation

$$y'' + P(x)\, y' + Q(x)\, y = 0$$

If P, Q are not well defined at $x = a$ then $x = a$ is a singular point.

1. <u>Regular Singular Point</u>
If $(x-a)P$ and $(x-a)^2 Q$ are not infinite at $x = a$, then $x = a$ is a regular singular point.

2. <u>Irregular Singular Point</u>
If $(x-a)P$ and $(x-a)^2 Q$ are infinite at $x = a$, then $x = a$ is an irregular singular point.

Note:- In some problems we can shift the origin to the point $x = c$, by putting $x = t + c$.

Solve the ODE $y'' + (x-1)^2 y' - 4(x-1)y = 0$ at $x = 1$

$$x = t + 1$$

$$\frac{dy}{dx} = \frac{dy}{dt}\cdot\frac{dt}{dx} = \frac{dy}{dt} \qquad \left[\frac{dt}{dx} = 1\right]$$

$$\frac{d^2y}{dx^2} = \frac{d^2y}{dt^2}$$

Hence the Equation becomes.

$$\frac{d^2y}{dt^2} + t^2\frac{dy}{dt} - 4ty = 0$$

Hence here, $t = 0$ is an ordinary point.

Let the solⁿ to be

$$y = \sum_{m=0}^{\infty} x^m a_m$$

$$y' = m \sum_{m=0}^{\infty} x^{m-1} a_m \quad , \quad y'' = m(m-1) \sum_{m=0}^{\infty} x^{m-2} a_m$$

Now on putting values,

$$m(m-1)\sum_{m=0}^{\infty} x^{m-2} a_m + m\sum_{m=0}^{\infty} x^{m+1} a_m - 4\sum_{m=0}^{\infty} x^{m+1} a_m = 0$$

i) Now equating to zero the coff^n of x^0

$$2(2-1)\,a_2 = 0$$

$$\boxed{a_2 = 0}$$

→ coff^n of x^1

$$3\times 2\, a_3 + 0 - 4a_0 = 0$$

$$\boxed{a_3 = \frac{2}{3} a_0}$$

→ coff^n of x^2

$$4\cdot 3\, a_4 + 1\cdot a_1 - 4a_1 = 0$$

$$\boxed{a_4 = \frac{a_1}{4}}$$

→ coff^n of x^3

$$5\cdot 4\, a_5 + 2a_2 - 4a_2 = 0$$

$$5\cdot 4\, a_5 - 2a_2 = 0$$

$$\therefore a_5 = \frac{2a_2}{5\cdot 4}$$

$$\boxed{a_5 = \frac{a_2}{10}} = 0$$

$$\boxed{a_5 = 0}$$

→ coff^n of x^r

$$(r+2)(r+1)\,a_{r+2} + (r-1)\,a_{r-1} - 4a_{r-1} = 0$$

$$\boxed{a_{r+2} = \frac{(r-5)\,a_{r-1}}{(r+1)(r+2)}}$$

i) $r = 5$

$$\boxed{a_7 = 0}$$

ii) $r = 6$

$$a_8 = \frac{a_5}{7\cdot 8} = 0$$

$$\boxed{a_8 = 0}$$

iii) $r = 7$

$$a_9 = \frac{2a_6}{8\cdot 9} = \frac{2a_6}{8\cdot 9} = \frac{-4a_0}{3\cdot 5\cdot 6\cdot 8\cdot 9}$$

iv) $r = 4$

$$a_6 = \frac{-1\,a_3}{5\cdot 6} = \frac{-2a_0}{3\cdot 5\cdot 6}$$

Thus, we get

$$y = a_0\left[1 + \frac{2}{3}(x-1)^3 + \frac{1}{45}(x-1)^6 - \frac{1}{1620}(x-1)^9 + \cdots\right]$$

$$+ a_1\left[(x-1) + \frac{(x-1)^4}{4} + \cdots\right]$$

* Find regular singular points of diffⁿ Eqⁿ

$$2x^2\frac{d^2y}{dx^2} + 3x\frac{dy}{dx} + (x^2-4)y = 0$$

we get

$$\frac{d^2y}{dx^2} + \frac{3}{2x}\frac{dy}{dx} + \left(\frac{x^2-4}{2x^2}\right)y = 0$$

$$P = \frac{3}{2x}, \quad Q = \frac{x^2-4}{2x^2}$$

at $x=0$ P, Q are infinite so $x=0$ is not a ordinary point.

$(x-0)P$ and $(x-0)^2Q$ are finite so $x=0$ is a regular singular Pt.

* Find regular singular points of the Diffⁿ Eqⁿ

$$x(x-2)^2y'' + 2(x-2)y' + (x+3)y = 0$$

we get

$$y'' + \frac{2\cancel{(x-2)}}{x\cancel{(x-2)}(x-2)}y' + \frac{(x+3)}{x(x-2)^2} = 0$$

$$y'' + \frac{2}{x(x-2)}y' + \frac{(x+3)}{x(x-2)^2} = 0$$

P and Q are infinite so $x=0$ and 2 are not ordinary Pt.

so $x = 0$ & 2 are the regular singular point.

Frobenius Method.

If $x=0$ is a regular singularity of Eqⁿ

$$y'' + P y' + Q y = 0$$

Then the series solution is given by.

$$y = \sum_{m=0}^{\infty} a_m x^{m+n} \quad \text{or} \quad x^n \sum_{m=0}^{\infty} a_m x^n$$

→ On Eqⁿ the coefficients of lowest power of x to zero, a quadratic Eqⁿ in m (indicial Equation) is obtained, and we will get two values of m.

Then following cases arise.

<u>Case I</u>: When two roots m_1, m_2 are different and not differing by an integer i.e $m_1 - m_2 \neq 0$, +ve integer.

{replace m by n here

Then complete solⁿ

$$y = a_0 \, y_{m_1} + b_0 \, y_{m_2}$$

<u>Case II</u>: When two roots m_1, m_2 are equal i.e, $m_1 = m_2$

Then Compⁿ Solⁿ

$$y = a_0 \, y_{m_1} + b_0 \left(\frac{\partial y}{\partial m}\right)_{m_1}$$

<u>Case III</u>: When roots m_1, m_2 are distinct and differ by an integer.

If some of the coefficient of y series of any m becomes infinite then replace a_0 by $b_0(m-m_1)$

$$a_0 = b_0(m - m_1)$$

complete solⁿ is

$$y = c_1 \, y_{m_1} + c_2 \left(\frac{\partial y}{\partial m}\right)_{m_1}$$

Case IV : When Roots of the eqn are distinct

* Find Soln in Generalized series about $x=0$

$$3x\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = 0 \quad \text{——— (1)}$$

we have
$$\frac{d^2y}{dx^2} + \frac{2}{3x}\frac{dy}{dx} + \frac{y}{3x} = 0$$

we clearly see that $x=0$ is a regular singular point

And let the soln to be

$$y = \sum_{m=0}^{\infty} a_m x^{m+n}$$

$$y' = \sum_{m=0}^{\infty} (m+n) a_m x^{m+n-1}$$

$$y'' = \sum_{m=0}^{\infty} (m+n)(m+n-1) a_m x^{m+n-2}$$

on putting in Eqn (1), we get

$$3(m+n)(m+n-1)\sum_{m=0}^{\infty} a_m x^{m+n-1} + 2(m+n)\sum_{m=0}^{\infty} a_m x^{m+n-1} + \sum_{m=0}^{\infty} a_m x^{m+n}$$

Now equating to zero the coffn of lowest power of x ie x^{m-1}

$$3m(m-1)a_0 + 2m a_0 = 0$$

$$3m^2 + 2m - 3m = 0 \qquad a_0 \neq 0$$

$$\boxed{m=0\ ,\ m=1/3}$$

Now on equating the coff^n of x^m

$$3(n+1)(n)a_1 + 2(n+1)a_1 + a_0 = 0$$

$$a_1[3n^2 + 3n + 2n + 1] = -a_0$$

$$\boxed{a_1 = -\frac{a_0}{(n+1)(3n+2)}}$$

Now equating to zero the coff^n of x^{m+n} to 0

$$3(m+n+1)(m+n)a_{m+1} + 2(m+n+1)a_{m+1} + a_m = 0$$

$$a_{m+1}[(m+n+1)(3m+3n+2)] + a_m = 0$$

$$\boxed{a_{m+1} = \frac{-a_m}{(m+n+1)(3m+3n+2)}}$$

i) for $m = 0$

$$a_1 = \frac{-a_0}{(n+1)(3n+2)}$$

ii) for $m = 1$

$$a_2 = \frac{-a_1}{(n+2)(3n+5)} = +\frac{a_0}{(n+1)(n+2)(3n+2)(3n+5)}$$

iii) for $m = 2$

$$a_3 = \frac{-a_2}{(n+3)(3n+8)} = \frac{-a_0}{(n+1)(n+2)(n+3)(3n+2)(3n+5)(3n+8)}$$

$\rightarrow$ At $n = 0$

$$a_1 = \frac{-a_0}{2}, \quad a_2 = \frac{a_0}{20}, \quad a_3 = \frac{-a_0}{480}$$

Thus at $n = 0$

$$y_1 = a_0\left[1 - \frac{1}{2}x + \frac{1}{20}x^2 - \frac{1}{480}x^3 + \cdots\right]$$

$\longrightarrow$ At $n = 1/3$

$a_1 = \frac{-a_0}{4}$, $a_2 = \frac{a_0}{56}$, $a_3 = \frac{-a_0}{1680}$

Thus at $n = 1/3$

$$y_2 = b_0\left(x^{1/3} - \frac{1}{4}x^{4/3} + \frac{1}{56}x^{7/3} - - - - -\right)$$

Hence the complete Solution will by

$$y = \left[a_0\left(1 - \frac{x}{2} + \frac{x^2}{20} - \frac{x^3}{480} + - -\right) + b_0 x^{1/3}\left(1 - \frac{x}{4} + \frac{x^2}{56} - - -\right)\right]$$

solved

Solve the power Series about $x = 0$

$$2x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} - (x+1)\,y = 0 \quad \text{———} \quad (1)$$

Let

$$y = \sum_{m=0}^{\infty} a_m x^{m+n}$$

$$y' = \sum_{m=0}^{\infty} (m+n)\, a_m x^{m+n-1}$$

$$y'' = \sum_{m=0}^{\infty} (m+n)(m+n-1)\, a_m x^{m+n-2}$$

on putting in eq^n (1), we get

$$2(m+n)(m+n-1)\sum_{m=0}^{\infty} a_m x^{m+n} + (m+n)\sum_{m=0}^{\infty} a_m x^{m+n} - \sum_{m=0}^{\infty} a_m x^{m+n+1}$$

$$- \sum_{m=0}^{\infty} a_m x^{m+n} = 0$$

Now eq^n to zero the coff^n to lowest power of x i.e x^n

$2n(n-1)\,a_0 + n\,a_0 - a_0 = 0$

$[2n(n-1) + (n-1)]\,a_0 = 0$ $\qquad a_0 \neq 0$

$(n-1)(2n+1)$

$\boxed{n = 1}$ $\boxed{n = -1/2}$

Now Eq^n to zero the coff^n of x^{n+1}

$$2(1+n)(n)\,a_1 + (1+n)\,a_1 - a_0 - a_1 = 0$$

$$a_1[\,2n(1+n) + (1+n) - 1\,] - a_0 = 0$$

$$a_1[\,2n + 2n^2 + n\,] = a_0$$

$$\boxed{a_1 = \frac{a_0}{n(2n+3)}}$$

Now equating to zero the coff^n of x^{m+n}.

$$[2(m+n)(m+n-1) + (n+m) - 1]\,a_m - a_{m-1} = 0$$

$$a_m\,[(m+n)[2m+2n-3]-1] = a_{m-1}$$

$$\boxed{a_m = \frac{a_{m-1}}{(m+n)(2m+2n-3)-1}}$$

Replace $m+1$ by m

$$a_{m+1} = \frac{a_m}{(m+n+1)[2m+2n+1]-1}$$

(i) $m = 1$

$$a_2 = \frac{a_1}{(n+2)(2n+3)-1}$$

(ii) $m = 2$

$$a_3 = \frac{a_2}{(3+n)(2n+5)-1}$$

(iii) $m = 3$

$$a_4 = \frac{a_3}{(n+4)(2n+7)-1}$$

$m = 1$	$n = -1/2$
$a_1 = \frac{a_0}{5}$	$a_1 = -a_0$
$a_2 = \frac{a_1}{14} = \frac{a_0}{70}$	$a_2 = \frac{-a_0}{2}$
$a_3 = \frac{a_2}{27} = \frac{a_0}{1890}$	$a_3 = \frac{a_2}{9} = \frac{a_0}{-18}$
$a_4 = \frac{a_3}{44} = \frac{a_0}{44 \times 1890}$	$a_4 = \frac{a_3}{20} = \frac{a_0}{-18 \times 20} = \frac{-a_0}{360}$

we have, $y = x^m (a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots)$

$\longrightarrow$ for $n = 1$

$$(y)_{n=1} = x\left(a_0 + \frac{a_0}{5}x + \frac{a_0}{70}x^2 + \frac{a_0}{1890}x^3 + \cdots\right)$$

$$y_1 = a_0 x\left(1 + \frac{x}{5} + \frac{x^2}{70} + \frac{x^3}{1890} + \cdots\right)$$

$\longrightarrow$ $n = -1/2$

$$(y)_{-1/2} = x^{-1/2}\left(a_0 + a_0 x - \frac{a_0}{2}x^2 - \frac{a_0}{18}x^3 \cdots\right)$$

$$= a_0 x^{-1/2}\left(1 - x - \frac{x^2}{2} - \frac{x^3}{18} \cdots\right)$$

Complete solution is given by.

$$y = Ax\left(1 + \frac{x}{5} + \frac{x^2}{70} + \frac{x^3}{1890} + \cdots\right)$$

$$+ Bx^{-1/2}\left(1 - x - \frac{x^2}{2} - \frac{x^3}{18} \cdots\right)$$

Solve $x(x-1)y'' + (3x-1)y' + y = 0$

Let $y = \sum_{m=0}^{\infty} a_m x^{m+n}$

$y' = (m+n) \sum_{m=0}^{\infty} a_m x^{m+n-1}$

$y'' = (m+n)(m+n-1) \sum_{m=0}^{\infty} a_m x^{m+n-2}$

on putting in eq^n (i) we get

$\left[(m+n)(m+n-1) + 3(m+n) + 1\right]\left(\sum_{m=0}^{\infty} a_m x^{m+n}\right) - (m+n)(m+n-1) \sum_{m=0}^{\infty} a_m x^{m+n-1}$

$+ (m+n) \sum_{m=0}^{\infty} a_m x^{m+n-1}$

Eq^n to zero the coff^n of x^{n-1}

$[n(n-1) + n]\, a_0 = 0$

$a_0 n(n) = 0$

$\boxed{n = n = 0}$

Eq^n to zero the coff^n of x^n

$[n(n-1) + 3n + 1]\, a_0 + [-(1+n)(n) \cancel{+} (1+n)]\, a_1 = 0$

$(n^2 + 2n + 1)\, a_0 - (1+n)^2 a_1 = 0$

$\boxed{a_1 = a_0}$

Now equating to zero the coff^n of x^{m+n}

$[(m+n)(m+n+2)+1]\, a_m - a_{m+1}[(m+n+1)(m+n) + (m+n+1)] = 0$

$[(m+n+1)^2]\, a_m - a_{m+1}[(m+n+1)(m+n+1)] = 0$

$\boxed{a_{m+1} = a_m}$

(i) $m = 0$

$\boxed{a_1 = a_0}$

(ii) $m = 1$

$\boxed{a_2 = a_1 = a_0}$

(iii) $m = 2$

$\boxed{a_3 = a_2 = a_0}$

- - - - - - -

we have

$$y = x^n(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \text{-----})$$

$$y = x^n(a_0 + a_0 x + a_0 x^2 + a_0 x^3 + \text{-----}) \quad (2)$$

$\rightarrow$ At $n = 0$

$$(y)_{n=0} = y_1 = a_0[1 + x + x^2 + x^3 + \text{-----}]$$

second solution is given by

$$\left(\frac{\partial y}{\partial m}\right)_{n=0}$$

on diff^n eq^n (2) wrt m

$$\frac{\partial y}{\partial m} = x^n \log x\, a_0 (1 + x + x^2 + x^3 + \text{-------})$$

$$y_2 = \left(\frac{\partial y}{\partial m}\right)_{n=0} = \log x\, a_0 (1 + x + x^2 + x^3 + \text{-----})$$

Complete sol^n is given by

$$y = c_1 y_1 + c_2 y_2$$

$$= \underbrace{c_1 a_0}_{A}[1 + x + x^2 \text{-----}] + \underbrace{c_2 a_0}_{B} \log x\, [1 + x + x^2 + \text{---}]$$

Solve in series the D.E

$$x y'' + y' - y = 0$$

Let the sol^n to be

$$y = \sum_{m=0}^{\infty} a_m x^{m+n}$$

$$y' = (m+n) \sum_{m=0}^{\infty} a_m x^{m+n-1}$$

$$y'' = (m+n)(m+n-1) \sum_{m=0}^{\infty} a_m x^{m+n-2}$$

$$\left[(m+n)(m+n-1) + (m+n)\right]\left(\sum_{m=0}^{\infty} a_m x^{m+n-1}\right) - \sum_{m=0}^{\infty} a_m x^{m+n} = 0$$

$$(m+n)^2 \sum_{m=0}^{\infty} a_m x^{m+n-1} - \sum_{m=0}^{\infty} a_m x^{m+n} = 0$$

Eq^n to zero the coeff^n of x^{n-1} to zero.

$$n^2 a_0 = 0 \qquad a_0 \neq 0$$

$$\boxed{n = n = 0}$$

then equating to zero the coeff^n of x^n

$$(1+n)^2 a_1 - a_0 = 0$$

$$a_1 = \frac{a_0}{(1+n)^2}$$

then equating to zero the coff^n of x^{m+n}

$$(m+n+1)^2 a_{m+1} - a_m = 0$$

$$a_{m+1} = \frac{a_m}{(1+m+n)^2}$$

ii) $m = 1$

$$a_2 = \frac{a_1}{(2+n)^2} = \frac{a_0}{(1+n)^2 (2+n)^2}$$

iii) $m = 2$

$$a_3 = \frac{a_0}{(1+n)^2 (2+n)^2 (3+n)^2}$$

iv) $m = 3$

$$a_4 = \frac{a_0}{(1+n)^2 (2+n)^2 (3+n)^2 (4+n)^2}$$

Thus,

$$y = a_0 x^n \left[1 + \frac{x}{(1+n)^2} + \frac{x^2}{(1+n)^2(2+n)^2} + \frac{x^3}{(1+n)^2(2+n)^2(3+n)^2} + \cdots \right] \quad (2)$$

$$y_1 = (y)_{n=0} = a_0 \left[1 + \frac{x}{1^2} + \frac{x^2}{(2!)^2} + \frac{x^3}{(3!)^2} \cdots \right]$$

Second sol^n is given by $diff^n$ eq (2)

$$\frac{\partial y}{\partial n} = a_0 x^n \log x \left[1 + \frac{x}{(1+n)^2} + \frac{x^2}{(1+n)^2(2+n)^2} + \frac{x^3}{(1+n)^2(2+n)^2(3+n)^2} + \cdots \right]$$

$$+ a_0 x^n \left[-\frac{2x}{(1+n)^3} - \frac{2x^2}{(1+n)^2(2+n)^2}\left[\frac{1}{1+n} + \frac{1}{2+n}\right] - \frac{2x^3}{(1+n)^2(2+n)^2(3+n)^2}\left[\frac{1}{1+n} + \frac{1}{2+n} + \frac{1}{3+n}\right] \cdots \right]$$

$$y_2 = \left(\frac{\partial y}{\partial n}\right)_{n=0} = a_0 \log x \left[1 + \frac{x}{(1!)^2} + \frac{x^2}{(2!)^2} + \frac{x^3}{(3!)^2} + \cdots \right]$$

$$+ a_0 \left[-\frac{2x}{(1!)^2} - \frac{2x^2}{(2!)^2}\left(1+\frac{1}{2}\right) - \frac{2x^3}{(3!)^2}\left(1+\frac{1}{2}+\frac{1}{3}\right) \cdots \right]$$

Hence the comple $soln^n$ is

$$y = c_1 y_1 + c_2 y_2$$

$$= \left[\underbrace{c_1 a_0}_{A} + \underbrace{c_2 a_0}_{B} \log x\right]\left[1 + \frac{x}{1!} + \frac{x^2}{(2!)^2} + \frac{x^3}{(3!)^2} + \cdots\right]$$

$$- 2\underbrace{c_2 a_0}_{B} \left[\frac{x}{1!} + \frac{x^2}{(2!)^2}\left(1+\frac{1}{2}\right) + \frac{x^3}{(3!)^2}\left(1+\frac{1}{2}+\frac{1}{3}\right) + \cdots \right]$$

Solve in Series Diffⁿ Eqⁿ.

$x^2 y'' + 5xy' + x^2 y = 0$

Let the solⁿ to be

$$y = \sum_{m=0}^{\infty} x^{m+n} a_m$$

$$y' = (m+n) \sum_{m=0}^{\infty} a_m x^{m+n-1}$$

$$y'' = (m+n)(m+n-1) \sum_{m=0}^{\infty} a_m x^{m+n-2}$$

$$\left[(m+n)(m+n-1) + 5(m+n)\right] \sum_{m=0}^{\infty} a_m x^{m+n} + \sum_{m=0}^{\infty} a_m x^{m+n+2} = 0$$

$$(m+n)(m+n+4) \sum_{m=0}^{\infty} a_m x^{m+n} + \sum_{m=0}^{\infty} a_m x^{m+n+2} = 0$$

i) Equating to zero the coffⁿ of x^n

$n(n+4) a_0 = 0$

$\boxed{n = 0 \;,\; n = -4}$

ii) Eqⁿ to zero the coffⁿ of x^{n+1}

$(1+n)(n+5) a_1 = 0$

$\boxed{a_1 = 0}$

iii) Eqⁿ to zero the coffⁿ of x^{n+2}

$(n+2)(n+6) a_2 + a_0 = 0$

$$\boxed{a_2 = \frac{-a_0}{(n+2)(n+6)}}$$

Eqⁿ to zero the coffⁿ of x^{m+n+2}

$(m+n+2)(m+n+6) a_{m+2} + a_m = 0$

$$a_{m+2} = \frac{-a_m}{(m+n+2)(m+n+6)}$$

i) $m = 0$

$$a_2 = \frac{-a_0}{(n+2)(n+6)}$$

ii) $m=1$

$$a_3 = \frac{-a_1}{(n+3)(n+7)} = 0$$

iii) $m=2$

$$a_4 = \frac{-a_2}{(n+4)(n+8)} = \frac{a_0}{(n+2)(n+4)(n+6)(n+8)}$$

$$a_5 = a_7 = a_9 = \cdots\cdots = 0$$

$$y = a_0 x^n \left[1 - \frac{x^2}{(n+2)(n+6)} - \frac{x^4}{(n+2)(n+4)(n+6)(n+8)} \cdots\cdots\right] \quad \text{——— (2)}$$

$$y_1 = (y)_{n=0} \left[1 - \frac{x^2}{2\cdot 6} - \frac{x^4}{2\cdot4\cdot6\cdot8} \cdots\cdots\right]$$

Now for $n=-4$, the coffecient becomes infinities.
To avoid this put $a_0 = b_0(n+4)$ in eq^n (2)

$$y = b_0 x^n \left[(n+4) - \frac{x^2(n+4)}{(n+2)(n+6)} - \frac{x^4}{(n+2)(n+6)(n+8)} \cdots\cdots\right]$$

$$\frac{\partial y}{\partial n} = b_0 x^n \log x \left[(n+4) - \frac{x^2(n+4)}{(n+2)(n+6)} - \frac{x^4}{(n+2)(n+6)(n+8)} \cdots\cdots\right]$$

$$+ b_0 x^n \left[1 - \frac{x^2\{(n+4)[2n+8] - [2n^2+8n+16]\}}{[2n^2+8n+16]^2} - \frac{x^4(3n^2+32n+76)}{(n^3+16n^2+76n+96)^2} + \cdots\cdots\right]$$

$$y_2 = \left(\frac{\partial y}{\partial n}\right)_{n=-4} = b_0 x^{-4} \log x \left[\frac{-x^4}{4\cdot2\cdot(-2)} - \frac{x^6}{16} + \cdots\right]$$

$$+ b_0 x^{-4} \left[1 + \frac{x^2}{4} - \frac{x^4}{4} + \cdots\cdots\right]$$

$$y_2 = b_0 x^{-4} \log x \left[-\frac{x^4}{16} - \frac{x^6}{16} - - - - \right] + b_0 x^{-4} \left[1 + \frac{x^2}{4} - \frac{x^4}{4} - - \right]$$

Thus the complete sol^n.

$$y = \underbrace{c_1 a_0}_{A} \left(1 - \frac{x^2}{12} + \frac{x^4}{384} - - - - \right) + \underbrace{c_2 b_0}_{B} x^{-4} \log x \left(-\frac{x^4}{16} - \frac{x^6}{16} - - - - - - \right)$$

$$+ \underbrace{c_2 b_0}_{B} x^{-4} \left(1 + \frac{x^2}{4} - \frac{x^4}{4} + - - - \right)$$

LEGENRRE'S FUNCTIONS

The diffⁿ Equation of the form

$$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + n(n+1)y = 0 \qquad \text{— ①}$$

is known as Legendre's diffⁿ Eqⁿ.

→ Let the solution be $y = \sum_{m=0}^{\infty} a_m x^{m+r}$

$$y' = (m+r)\sum_{m=0}^{\infty} a_m x^{m+r-1}$$

$$y'' = (m+r)(m+r-1)\sum_{m=0}^{\infty} a_m x^{m+r-2}$$

Putting these values in eqⁿ ①, we get

$$(m+r)(m+r-1)\sum_{m=0}^{\infty} a_m x^{m+r-2} - 2(m+r)\sum_{m=0} a_m x^{m+r}$$

$$+ \left[n(n+1) - (m+r)(m+r-1)\right]\sum_{m=0}^{\infty} a_m x^{m+r}$$

Eqⁿ to zero the coffⁿ of x^{r-2}

$$\cancel{r(r-1)a_0 - 2r a_1 = 0}$$

$$\boxed{a_0 \neq 0} \quad \cancel{\Rightarrow r=0, r=1}$$

$$r(r-1)a_0 = 0$$

$$\boxed{r = 0, 1}$$

Eqⁿ to zero the coffⁿ of x^{r-1}

$$(1+r)(r)a_1 = 0$$

· Coffⁿ of x^{m+r}

$$(m+r+2)(m+r+1)\,a_{m+2} - 2(m+r)\,a_m - \left((m+r)(m+r-1) - n(n+1)\right) a_m = 0$$

$$a_{m+2} = \frac{a_m\left(2(m+r) + (m+r)(m+r-1) - n(n+1)\right)}{(m+r+1)(m+r+2)}$$

$$\boxed{a_{m+2} = \frac{a_m\left[(m+r)(m+r+1) - n(n+1)\right]}{(m+r+1)(m+r+2)}}$$

(5)

<u>Case I</u> $r = 0$

$$a_{m+2} = \frac{a_m\left[m(m+1) - n(n+1)\right]}{(m+1)(m+2)}$$

$\cancel{a_{m+2}} \Rightarrow$

$$a_2 = \frac{a_0\left[-n(n+1)\right]}{2!}$$

$$a_3 = \frac{a_1\left[2 - n(n+1)\right]}{3!} = \frac{(n+1)(n+2)\,a_1}{3!}$$

$$a_4 = \frac{a_2\left[6 - n(n+1)\right]}{3\cdot 4} \quad \cancel{\frac{a_0\left[-n(n+1)\right]}{4!}}$$

$$= \frac{\left[6 - n(n+1)\right]\left[-n(n+1)\right]c_0}{4!}$$

$$= \frac{\left[n^2 + n - 6\right]\left[n(n+1)\right]c_0}{4!}$$

$$= \frac{n(n+1)(n+2)(n-3)\,c_0}{4!}$$

Hence the requies solution is given as

$$y = \sum_{m=0}^{\infty} a_m x^{m+r}$$

$$y = x^r (a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots)$$

$$y = c_0 \left[1 - \frac{n(n+1)}{2!} x^2 + \frac{n(n+1)(n-2)(n+3)}{4!} x^3 \cdots \right.$$

$$+ c_1 \left[x - \frac{(n-1)(n+2)}{3!} x^3 + \frac{(n-1)(n-3)(n+2)(n+4)}{5!} x^5 \right.$$

$$+ \cdots \Big] \quad \text{——— (6)}$$

Since this solution itself is a solution with two arbitary constants. It is the most genealal solution of the Legendeeds differential Equation.

It is clear from eq^n (6) that for an even integer $n \geq 0$ the first of two series terminates and gives us a polynomial sol^n.

→ Legendre's polynomial of first kind is a series sol^n to the Legendre's Equation which are termanating.

Legendres Polynomial of first kind

Legendre's Polynomial is given by the series sol^n when the terms are terminating.

if $m = n$

from relation (5) $\quad a_{n+2} = a_{n+4} = a_{n+6} = 0$

$$c_m = \frac{(m+1)(m+2) - c_{m+2}}{m(m+1) - \cancel{n}(n+1)}$$

for $m = n-2, n-4$

$$C_{n-2} = \frac{n(n-1)\,C_n}{(n-2)(n-1) - n(n+1)} = -\frac{n(n-1)\,C_n}{2(2n-1)}$$

$$C_{n-4} = \cancel{\frac{-n(n-1)\,C_n}{2 \cdot 4 \cdot 2(n-1)(2n-3)}} \quad \frac{n(n-1)(n-2)(n-3)}{2 \cdot 4 (2n-1)(2n-3)}\, C_n$$

and so on.

This will lead to a sol^n. $y = \sum C_m x^{n+m}$

$$y = C_n \left[x^n - \frac{n(n-1)}{2(2n-1)} x^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2 \cdot 4 (2n-1)(2n-3)} x^{n-4} + \cdots \right]$$

Let C_n be $\dfrac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{n!}$ to get $P_n(1) = 1$

Therefore,

$$P_n(x) = \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{n!} \left[x^n - \frac{n(n-1)}{2(2n-1)} x^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2 \cdot 4 \cdot (2n-1)(2n-3)} x^{n-4} - \cdots \right]$$

$$P_n(x) = \sum_{r=0}^{n/2} \frac{(-1)^r (2n-2r)!}{2^n\, r!\, (n-2r)!\, (n-r)!}$$

Legendre's function of Second kind.

The legendre is fun^n of Second kind are the series sol^n to the Legendre's sol^n, which do not terminate.

The series which do not terminate when n is even.

1-

Generating funⁿ of Legⁿ Polyⁿ.

Prove that the $P_n(x)$ is the cofficient of t^n in the expansion of $(1-2xt+t^2)^{-1/2}$ in ascending power of x.

$$\left[1-t(2x-t)\right]^{-1/2} = \sum_{n=0}^{\infty} t^n P_n(x)$$

using,

$$(1-x)^{-n} = 1+nx+\frac{n(n+1)}{2!}x^2+\frac{n(n+1)(n+2)}{3!}x^3+\ldots$$

Here $x = t(2x-t)$, $n = 1/2$

$$\Rightarrow 1+\frac{1}{2}t(2x-t)+\frac{\frac{1}{2}\left(\frac{1}{2}+1\right)}{2!}\left[t(2x-t)\right]^2+\ldots$$

$$\Rightarrow 1+\frac{1}{2}t(2x-t)+\frac{1\cdot 3}{2\cdot 4}t^2(2x-t)^2+\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}t^3(2x-t)^3\ldots$$

$$\ldots+\frac{1\cdot 3\cdot 5\ldots(2n-5)}{2\cdot 4\cdot 6\ldots(2n-4)}t^{n-2}(2x-t)^{n-2}+$$

$$\frac{1\cdot 3\cdot 5\ldots(2n-3)}{2\cdot 4\cdot 6\ldots(2n-2)}t^{n-1}(2x-t)^{n-1}+\frac{1\cdot 3\cdot 5\ldots(2n-1)}{2\cdot 4\cdot 6\ldots 2n}t^n(2x-t)^n$$

using $$(x-a)^n = \sum_{r=0}^{\infty}(-1)^r\, {}^nC_r\, x^{n-r}a^r$$

→ To Expand last term.

$$\frac{1\cdot 3\cdot 5\ldots(2n-1)}{2\cdot 4\cdot 6\ldots 2n}t^n\left[\sum_{r=0}^{\infty}(-1)^r\, {}^nC_r\,(2x)^{n-r}t^r\right]$$

$$\frac{1\cdot 3\cdot 5 \cdots (2n-1)}{2\cdot 4\cdot 6 \cdots 2n} t^n \left[{}^nC_0 (2x)^n t^0 - {}^nC_1 (2x)^{n-1} t^1 \cdots \right]$$

$$\frac{1\cdot 3\cdot 5 \cdots (2n-1)}{2\cdot 4\cdot 6 \cdots 2n} {}^nC_0 (2x)^n$$ is the coff^n of t^n.

→ ~~To expand~~ ~~second last term~~

$$\frac{1\cdot 3\cdot 5 \cdots (2n-1)\; 2^n x^n}{2^n[1\cdot 2\cdot 3 \cdots n]} \qquad [{}^nC_0 = 1]$$

$$\frac{1\cdot 3\cdot 5 \cdots (2n-1)\; x^n}{n!} \quad \text{——— (1)}$$

→ To Expand Second last term

$$\frac{1\cdot 3\cdot 5 \cdots (2n-3)}{2\cdot 4\cdot 6 \cdots (2n-2)} t^{n-1} \left[\sum_{r=0}^{\infty} {}^{n-1}C_r (-1)^r (2x)^{n-r-1} t^r \right]$$

$$\frac{1\cdot 3\cdot 5 \cdots (2n-3)}{2\cdot 4\cdot 6 \cdots (2n-2)} t^{n-1} \left[{}^{n-1}C_0 (2x)^{n-1} t^0 - {}^{n-1}C_1 (2x)^{n-2} t^2 + \cdots \right]$$

$$\frac{1\cdot 3\cdot 5 \cdots (2n-3)}{2\cdot 4\cdot 6 \cdots (2n-2)} \left[- {}^{n-1}C_1\, 2^{n-2} x^{n-2} \right]$$ is the coff^n of t^n

$$\frac{1\cdot 3\cdot 5 \cdots (2n-3)}{2\cdot 4\cdot 6 \cdots (2n-2)} \left[-(n-1)\, 2^{n-2} x^{n-2} \right] \qquad [{}^{n-1}C_1 = (n-1)]$$

$$\frac{1\cdot 3\cdot 5 \cdots (2n-3)}{2\cdot \cancel{2^{n-2}} (n-1)!} \left[-(n-1)\, \cancel{2^{n-2}} x^{n-2} \right]$$

$$\frac{1\cdot 3\cdot 5 \cdots (2n-3)(2n-1)}{n!} \left[- \frac{n(n-1)}{2(2n-1)} x^{n-2} \right]$$

$$\frac{1\cdot3\cdot5\ldots\ldots(2n-1)}{n!}\left\{-\frac{n(n-1)}{2(2n-1)}x^{n-2}\right\} \quad \text{——— ②}$$

Similarly the coff^n of third last term.

$$\frac{1\cdot3\cdot5\ldots\ldots 2n-1}{n!}\left[\frac{n(n-1)(n-2)(n-3)}{2\cdot4\ldots\ldots(2n-1)(2n-3)}x^{n-4}\right] \quad \text{——— ③}$$

Thus the coff^n of t^n in the Expansion is the sum of all eq^n ①, ② & ③ ---- so on.

$$\frac{1\cdot3\cdot5\ldots\ldots(2n-1)}{n!}\left[x^n - \frac{n(n-1)}{2(2n-1)}x^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2\cdot4\,(2n-1)(2n-3)}x^{n-4} \ldots\ldots\ldots\right]$$

$= P_n(x)$ ie the Legendre's polynomial.

Thus coff^n of t, t^2, t^3 ------- are $P_1(x)$, $P_2(x)$, $P_3(x)$ -------

Hence.

$$(1-2xt+t^2)^{-1/2} = P_0(x) + tP_1(x) + t^2P_2(x) \ldots\ldots$$

$$= \sum_{n=0}^{\infty} P_n(x)\, t^n$$

Proved

① P.t $P_n(1) = 1$

we have

$$(1 - 2tx + t^2)^{-1/2} = \sum_{n=0}^{\infty} P_n(x)\, t^n$$

$$(1 - 2t + t^2)^{-1/2} = \sum_{n=0}^{\infty} P_n(1)\, t^n$$

$$(1 - t)^{-1} = \sum_{n=0}^{\infty} P_n(1)\, t^n$$

$$1 + t + t^2 + \text{------} t^n \text{---} = \sum_{n=0}^{\infty} P_n(1)\, t^n$$

on Equating we get

$$\boxed{P_n(1) = 1}$$ Proved

② $P_n(-x) = (-1)^n P_n(x)$

we have

$$(1 - 2tx + t^2)^{-1/2} = \sum_{n=0}^{\infty} P_n(x)\, t^n \quad \text{———— (A)}$$

Put $x = -x$ in eq^n (A)

$$(1 + 2tx + t^2)^{-1/2} = \sum_{n=0}^{\infty} P_n(-x)\, t^n \quad \text{———— (1)}$$

Put $t = -t$ in eq^n (A)

$$(1 + 2tx + t^2)^{-1/2} = \sum_{n=0}^{\infty} P_n(x)\, (-t)^n \quad \text{———— (2)}$$

from (I) & (II)

$$\sum_{n=0}^{\infty} P_n(-x)\, t^n = \sum_{n=0}^{\infty} P_n(x)\,(-1)^n (t)^n$$

on Equating, we get,

$$\boxed{P_n(-x) = (-1)^n P_n(x)}$$ Proved.

for $x=1$

$$P_n(-1) = (-1)^n P(1)$$

$$\boxed{P_n(-1) = (-1)^n}$$

P.T :- i) $P'_n(1) = \frac{1}{2} n(n+1)$

ii) $P'_n(-1) = (-1)^{n-1} \frac{n}{2}(n+1)$

Sol^n Legendre's Eq^n is

$$(1-x^2)\, y'' - 2x\, y' + n(n+1)\, y = 0$$

we know that $P_n(x)$ is the Sol^n of this Eq^n

i) Put $x=1$

$$0 - 2P'_n(1) + n(n+1)\, P_n(1) = 0$$

$$P'_n(1) = \frac{1}{2} n(n+1)$$

Proved

ii) Putting $x=-1$

$$0 - 2P'_n(-1) + n(n+1)\, P_n(-1) = 0$$

$$P'_n(-1) = (-1)^n \frac{n(n+1)}{2}$$

Proved

Rodrigue's formula

The $P_n(x)$ is given as

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2-1)^n$$

Proof:- Let $(x^2-1)^n = u$

$$\frac{du}{dx} = u_1 = n(x^2-1)^{n-1} 2x \qquad \text{[on diff}^n$$

$$(x^2-1) u_1 = n(x^2-1)^n 2x$$

$$(x^2-1) u_1 = 2nxu \qquad \text{[Again diff}^n]$$

$$(x^2-1) u_2 + u_1 2x = 2n(xu_1 + u)$$

using libertz's theorem, we have

$$\frac{d^n}{dx^n}(uv) = u(V)_n + {}^nC_1 (u)_1 (V)_{n-1} + {}^nC_2 (u)_2 (V)_{n-2} - - - -$$

$$- - - - - - (u)_n (V)$$

on diff^n n times using libertz's theorem.

$$\left[(x^2-1) u_{n+2} + {}^nC_1 (2x) u_{n+1} + {}^nC_2 (2) U_n\right] + 2\left[x u_{n+1} + {}^n C_1 u_n\right]$$

$$= 2n\left[x u_{n+1} + {}^nC_1 U_n\right] + 2n U_n$$

$$(x^2-1) U_{n+2} + \left[2nx U_{n+1} - 2x U_{n+1} - 2nx U_{n+1}\right]$$

$$+ \left[(n\, n+1) U_n + 2n U_n - 2n^2 U_n + 2n U_n\right) = 0$$

$$(x^2-1) U_{n+2} + 2x U_{n+1} + n(n+1) U_n = 0$$

$$(x^2-1) U_n'' + 2x U_n' - n(n+1) U_n = 0$$

This is the legendary diffth^n equation with U_n as its

Solution

we know $P_n(x)$ is the sol^n of Legendary Eq^n.

Let $P_n(x) = K\ U_n(x)$

$$= K \frac{d^n}{dx^n}[(x^2-1)^n] \quad \text{—— (A)}$$

$$= K \frac{d^n}{dx^n}[(x-1)^n (x+1)^n]$$

Now again applying Libnitz's theorem.

$$= K\Big[(x-1)^n [(x+1)^n]_n + {}^nC_1 [n(x-1)^{n-1}][(x+1)^n]_{n-1} \text{ ------ }$$

$$\text{---------- } {}^nC_n [(x-1)^n]_n (x+1)^n\Big] \quad \text{—— (1)}$$

if $Z = (x-1)^n$

$Z_1 = n(x-1)^{n-1}$

$Z_2 = n(n-1)(x-1)^{n-2}$

$Z_n = n(n-1)(n-2) \text{ ---- } 3.2.1\ (x-1)^{n-n}$

$= n(n-1)(n-2) \text{ ---- } 3.2.1$

$[(x-1)^n]_n = n!$

Thus from eq^n (1)

At $x = 1$

$P_n(1) = K[0 + 0 + \text{ ---------- } + 1.\ n!\ 2^n]$

$[P_n(1) = 1]$

$1 = K\ n!\ 2^n$

$K = \frac{1}{n!\ 2^n}$ —— (2)

from eq^n ② & ④, we get

$$\boxed{P_n(x) = \frac{1}{n!\,2^n} \frac{d^n}{dx^n}\left[(x^2-1)^n\right]}$$

* S.t $\int_{-1}^{+1} P_n(x)\,dx = 0$, $n \neq 0$

$\int_{-1}^{+1} P_n(x)\,dx = 2$, $n = 0$

Sol^n we have

$$P_n(x) = \frac{1}{n!\,2^n} \frac{d^n}{dx^n}\left[(x^2-1)^n\right]$$

$$\int_{-1}^{1} \frac{1}{n!\,2^n} \frac{d^n}{dx^n} (x^2-1)^n \, dx$$

$$= \frac{1}{n!\,2^n} \left[\frac{d^{n-1}}{dx^{n-1}} (x^2-1)^n\right]_{-1}^{1}$$

$$= 0$$

ii) if $n = 0$

$$\int_{-1}^{1} P_0(x)\,dx = \int_{-1}^{1} 1\,dx$$

$$= [x]_{-1}^{1}$$

$$= 2 \quad \text{Proved}$$

Legendre's Polynomial

using Rodrigue's formula we have,

$$P_n(x) = \frac{1}{n!\,2^n}\frac{d^n}{dx^n}\left[(x^2-1)^n\right]$$

(i) $n=0$

$$P_0(x) = \frac{1}{2^0\,1!} = 1 = P_0(x)$$

ii) $n=1$

$$P_1(x) = \frac{1}{1!\,2^1}\frac{d}{dx}(x^2-1) = x = P_1 x$$

iii) $n=2$

$$P_2(x) = \frac{1}{2!\,2^2}\frac{d^2}{dx^2}(x^2-1)^2 = \frac{d}{dx}\frac{1}{8}\left[2\cdot(x^2-1)\cdot 2x\right]$$

$$= \frac{1}{2}\left[(x^2-1) + 2x^2\right]$$

$$= \frac{1}{2}(3x^2-1) = P_2(x)$$

Similarly

$$P_3(x) = \frac{1}{2}(5x^3 - 3x)$$

$$P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3)$$

$$P_5(x) = \frac{1}{8}(63x^5 - 70x^3 + 15x)$$

$$P_6(x) = \frac{1}{16}(231x^6 - 315x^4 + 105x^2 - 5)$$

$$\vdots \qquad\qquad \vdots \qquad\qquad \vdots$$

and so on.

$$P_n(x) = \sum_{r=0}^{n/2} \frac{(-1)^r\,(2n-2r)!}{2^n\,r!\,(n-r)!\,(n-2r)!}\,x^{n-2r}$$

where $n/2 = \begin{cases} n/2, & \text{if } n \text{ even} \\ (n-1)/2, & \text{if } n \text{ odd.} \end{cases}$

Expand $f(x) = 4x^3 - 2x^2 - 3x + 8$ in terms of Legendre polynomials.

Sol^n we have $P_3(x) = \frac{1}{2}(5x^3 - 3x)$

$P_2(x) = \frac{1}{2}(3x^2 - 1)$

$P_1(x) = x$

$P_0(x) = 1$

Now consider

$$4x^3 - 2x^2 - 3x + 8 = A(P_3(x)) + B(P_2(x)) + C(P_1(x)) + D(P_0(x))$$

$$4x^3 - 2x^2 - 3x + 8 = A\frac{5}{2}x^3 - A\frac{3}{2}x + B\frac{3}{2}x^2 - \frac{B}{2} + Cx + D$$

on comparing we get

$4 = \frac{5A}{2}$

$\boxed{A = 8/5}$

$\frac{3}{2}B = -2$

$\boxed{B = -\frac{4}{3}}$

$-\frac{B}{2} + D = 8$

$D = 8 + \left(-\frac{4}{2 \times 3}\right)$

$\boxed{D = \frac{22}{3}}$

$-\frac{3}{2}A + Cx = -3x$

$\boxed{C = -3} + \frac{12}{5}$

$\boxed{C = -3/5}$

we get

$$f(x) = \frac{8}{5}P_3(x) - \frac{4}{3}P_2(x) - \frac{3}{5}P_1(x) + \frac{22}{3}P_0(x).$$

Recurrence Relations

① $(2n+1)x\,P_n(x) = (n+1)P_{n+1}(x) + nP_{n-1}(x)$

we have from defⁿ of Genⁿ funⁿ

$$\frac{1}{\sqrt{1-2xt+t^2}} = \sum_{n=0}^{\infty} t^n P_n(x) \quad \text{——} ①$$

on diffⁿ w.r.t. t, we get

$$\frac{-(2t-2x)}{2(1-2xt+t^2)^{3/2}} = \sum_{n=0}^{\infty} n t^{n-1} P_n(x)$$

$$\frac{x-t}{(1-2xt+t^2)^1} \frac{1}{(1-2xt+t^2)^{1/2}} = \sum_{n=0}^{\infty} n t^{n-1} P_n(x)$$

from eqⁿ ①

$$(x-t)\sum_{n=0}^{\infty} t^n P_n x = (1-2xt+t^2)\sum_{n=0}^{\infty} n t^{n-1} P_n(x)$$

Now eqⁿ the coffⁿ of t^n on both sides,

$$x P_n(x) - P_{n-1}(x) = (n+1)P_{n+1}(x) - 2xnP_n(x) + (n-1)P_{n-1}(x)$$

$$\boxed{(2n+1)x\,P_n(x) = (n+1)P_{n+1}(x) + nP_{n-1}(x)}$$

Proved

→ if $n \to n-1$

$$\boxed{(2n-1)x\,P_{n-1}(x) = nP_n(x) + (n-1)P_{n-2}(x)}$$

Proved

② $n P_n(x) = x P_n'(x) - P_{n-1}'(x)$

we have

$$\frac{1}{(1-2xt+t^2)^{1/2}} = \sum_{n=0}^{\infty} t^n P_n(x)$$

diffⁿ w.r.t. t we get

$$-\frac{1}{2}(1-2xt+t^2)^{-3/2}(-2x+2t) = \sum_{n=0}^{\infty} n t^{n-1} P_n(x)$$

$$-\frac{t-x}{(1-2xt+t^2)} \cdot \frac{1}{\sqrt{1-2xt+t^2}} = \sum_{n=0}^{\infty} n t^{n-1} P_n(x)$$

$$-(t-x)(1-2xt+t^2)^{-3/2} = \sum_{n=0}^{\infty} n t^{n-1} P_n(x) \qquad \text{---} (I)$$

diffⁿ again w.r.t. x, we get

$$-\frac{1}{2}(1-2xt+t^2)^{-3/2}(-2t) = \sum_{n=0}^{\infty} t^n P_n'(x)$$

$$t(1-2xt+t^2)^{-3/2} = \sum_{n=0}^{\infty} t^n P_n'(x) \qquad \text{---} (II)$$

divide eq (I) by (II)

$$\frac{x-t}{t} = \frac{\sum_{n=0}^{\infty} n t^{n-1} P_n(x)}{\sum_{n=0}^{\infty} t^n P_n'(x)}$$

$$(x-t)\sum_{n=0}^{\infty} t^n P'_n(x) = t\sum_{n=0}^{\infty} n t^{n-1} P_n(x)$$

Now eqⁿ to zero the coffⁿ of t^n

$$x P'_n(x) - P'_{n-1}(x) = n P_n(x)$$

$$\boxed{n P_n(x) = x P'_n(x) - P'_{n-1}(x)}$$

Proved

③ $(2n+1) P_n(x) = P'_{n+1}(x) - P'_{n-1}(x)$

we have $(2n+1)x P_n(x) = (n+1) P_{n+1}(x) + n P_{n-1}(x)$

on diff ① w.r.t x we get

$(2n+1) P_n(x) + (2n+1)x P'_n(x) = (n+1) P'_{n+1}(x) + n P'_{n-1}(x)$

from Relation ⑪ $\quad x P'_n(x) = n P_n(x) + P'_{n-1}(x)$

$(2n+1)P_n(x) + (2n+1) n P_n(x) + (2n+1) P'_{n-1}(x) = (n+1) P'_{n+1}(x) + n P'_{n-1}(x)$

$(2n+1)(n+1) P_n(x) + (n+1)P'_{n-1}(x) = (n+1) P'_{n+1}(x)$

we get

$$\boxed{(2n+1) P_n(x) = P'_{n+1}(x) - P'_{n-1}(x)}$$

Proved

④ $(n+1) P_n(x) = P'_{n+1}(x) - x P'_n(x)$

Subtraction Relation ⑪ from ⑪

$(2n+1-n) P_n(x) = P'_{n+1}(x) - P'_{n-1}(x) - x P'_n(x) + P'_{n-1}(x)$

$$\boxed{(n+1) P_n(x) = P'_{n+1}(x) - x P'_n(x)}$$

Proved

⑤ $(1-x^2) P'_n(x) = n [P_{n-1}(x) - x P_n(x)]$

we have

$(n+1) P_n(x) = P'_{n+1}(x) - x P'_n(x)$

Replacing $n+1$ by n, we get

$n P_{n-1}(x) = P'_n(x) - x P'_{n-1}(x)$ ——— ①

from Relation ⑪ we get

$n P_n(x) = x P'_n(x) - P'_{n-1}(x)$ ——— ⑪

on multiplying eq (II) by x & sub. it from eq^n (I)

$$n P_{n-1}(x) - x n P_n(x) = P'_n(x) - x^2 P'_n(x) - x P'_{n-1}(x) + x P'_{n-1}(x)$$

$$n[P_{n-1}(x) - x P_n(x)] = (1-x^2) P'_n(x)$$

Thus,

$$\boxed{(1-x^2) P'_n(x) = n\,[P_{n-1}(x) - x P_n(x)]}$$

(6) $(x^2-1) P'_n = (n+1)\,[-x P_n(x) + P_{n+1}]$

we have from Relation (1)

$$(2n+1) x P_n(x) = (n+1) P_{n+1}(x) + n P_{n-1}(x)$$

$$(n+1)(P_{n+1} - x P_n) = n(x P_n - P_{n-1}) \quad \text{——— (I)}$$

we have

$$(x^2-1) P'_n(x) = n(x P_n - P_{n-1}) \quad (5) \quad \text{——— (II)}$$

from (I) & (II), we get

$$\boxed{(x^2-1) P'_n(x) = (n+1)\,[P_{n+1}(x) - x P_n]}$$

(7) Beltrami's Result

$$(2n+1)(x^2-1) P'_n(x) = n(n+1)[P_{n+1}(x) - P_{n-1}(x)]$$

from Relation (6)

$$(x^2-1) P'_n(x) = (n+1)[P_{n+1}(x) - x P_n(x)] \quad \text{——— (I)}$$

from Relation (5)

$$(x^2-1) P'_n(x) = n[x P_n(x) - P_{n-1}(x)] \quad \text{——— (II)}$$

Multiply eq (I) by n & (II) by $(n+1)$, we get

$(n+n+1)(x^2-1)P'_n(x) = n(n+1)P_{n+1}(x) + n(n+1)P_{n-1}(x)$

$$\boxed{(2n+1)(x^2-1)P'_n(x) = n(n+1)\left[P_{n+1}(x) - P_{n-1}(x)\right]}$$

Proved

Orthogonal property of Legn Polyn

If $P_m(x)$ and $P_n(x)$ are Legendre's polynomial (m, n +ve integers), then,

$$\int_{-1}^{1} P_m(x)P_n(x)\,dx = \begin{cases} 0, & \text{if } m \neq n \\ \frac{2}{2n+1}\delta_{mn}, & \text{if } m = n \end{cases}$$

Proof:-

Case I $m \neq n$

As $P_m(x)$ and $P_n(x)$ are the Legendre's Polynomial, they satisfy the Legendre's Eqn. Therefore, we get

$$(1-x^2)P''_m(x) - 2x P'_m(x) + m(m+1)P_m = 0 \quad \text{——} (1)$$

$$(1-x^2)P''_n - 2x P'_n(x) + n(n+1)P_n = 0 \quad \text{——} (11)$$

Multiply eqn (1) by P_n and (2) by P_m, the subtra-

$$(1-x^2)(P_nP''_m - P_mP''_n) - 2x(P_nP'_m - P_mP'_n) + [m(m+1) - n(n+1)]P_nP_m = 0 \quad \text{——} (3)$$

$$(1-x^2)\frac{d}{dx}(P_nP'_m - P_mP'_n) - 2x(P_nP'_m - P_mP'_n) = (n^2 - m^2 + n - m)P_nP_m \quad \text{——} (4)$$

$$\frac{d}{dx}\left[(1-x^2)(P_nP'_m - P_mP'_n)\right] = (n-m)(n+m+1)P_nP_m \quad \text{——} (5)$$

on Integrating both sides from -1 to 1 w.r.t x, we get

$$(n-m)(n+m+1)\int_{-1}^{1} P_m(x)P_n(x)\,dx$$

$$= \Big[(1-x^2)\,(P_n P_m' - P_m P_n')\Big]_{x=-1}^{1}$$

Therefore $\boxed{\int_{-1}^{+1} P_m(x)\,P_n(x)\,dx = 0}$ if $m \neq n$

Case II $\quad m = n$

using def^n of Generating fun^n for Leg fun^n.

$$(1-2xt+t^2)^{-1/2} = \sum_{n=0}^{\infty} t^n P_n(x) \quad \text{——— (A)}$$

Also,

$$(1-2xt+t^2)^{-1/2} = \sum_{m=0}^{\infty} t^m P_m(x) \quad \text{——— (B)}$$

on Multiplying Eq^n (A), (B) we get

$$(1-2xt+t^2)^{-1} = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} t^{m+n} P_m(x)\,P_n(x)$$

Now on integrating both sides w.r.t x from -1 to 1 Put $n=m$

$$\sum_{n=0}^{\infty}\left[\int_{-1}^{1} P_n(x)^2\,dx\right] t^{2n} = \int_{-1}^{1}\frac{dx}{1-2xt+t^2}$$

$$= \left[\frac{\ln(1-2xt+t^2)}{-2t}\right]_{-1}^{1}$$

$$= \frac{-1}{2t}\left[\ln(1-t)^2 - \ln(1+t)^2\right]$$

$$= \frac{-1}{2t}2\left[\ln(1-t) - \ln(1+t)\right]$$

$$= \frac{1}{t}\left[\ln(1+t) - \ln(1-t)\right]$$

$$= \frac{1}{t}\left[\left(t - \frac{t^2}{2} + \frac{t^3}{3} ----\right) - \left(-t - \frac{t^2}{2} - \frac{t^3}{3} ---\right)\right]$$

$$= \frac{2}{t}\left[t + \frac{t^3}{3} + \frac{t^5}{5} ------\right]$$

$$= \frac{2}{t}\left[t + \frac{t^3}{3} + \frac{t^5}{5} ------\right]$$

$$= \frac{2}{t}\sum_{n=0}^{\infty}\frac{t^{2n+1}}{2n+1}$$

$$= 2\sum_{n=0}^{\infty}\frac{t^{2n}}{2n+1}$$

$$\boxed{\int_{-1}^{1}(P_n(x))^2\,dx = \frac{2}{2n+1}}$$

on combinging both case we get

$$\int_{-1}^{+1} P_n(x)\,P_m(x)\,dx = \frac{2}{2n+1}\delta_{mn}$$

Bessel's functions.

The diffⁿ Eqⁿ

$$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2-n^2)y = 0 \quad \text{———} \text{(1)}$$

called as Bessel's diffⁿ Equation.

→ consider the solⁿ of this Eqⁿ be

$$y = \sum_{m=0}^{\infty} a_m x^{m+r}$$

$$y' = (m+r) \sum_{m=0}^{\infty} a_m x^{m+r-1}$$

$$y'' = (m+r)(m+r-1) \sum_{m=0}^{\infty} a_m x^{m+r-2}$$

putting in Eqⁿ (1), we get

$$0 = \left[(m+r)(m+r-1) + (m+r) - n^2\right] \sum_{m=0}^{\infty} a_m x^{m+r}$$

$$+ \sum_{m=0}^{\infty} a_m x^{m+r+2} = 0$$

$$= \left[(m+r)^2 - n^2\right] \sum_{m=0}^{\infty} a_m x^{m+r} + \sum_{m=0}^{\infty} a_m x^{m+r+2} = 0$$

i) coffⁿ of x^r

$$(r^2-n^2)\, a_0 = 0 \qquad a_0 \neq 0$$

$$\boxed{r = n, -n}$$

ii) coffⁿ of x^{r+1}

$$\left[(1+r)^2 - n^2\right] a_1 = 0 \qquad \because (1+r)^2 - n^2 \neq 0$$

so $\boxed{a_1 = 0}$

Coeffⁿ of x^{m+r+2}

$$[(m+r+2)^2-n^2]\,a_{m+2} + a_m = 0$$

$$a_{m+2} = \frac{-a_m}{[(m+r+2)^2-n^2]}$$

$$a_3 = \frac{-a_1}{(r+3)^2-n^2} = 0$$

$$a_1 = a_3 = a_5 = a_7 = 0$$

i) $m=0$

$$a_2 = \frac{-a_0}{[(r+2)^2-n^2]}$$

ii) $m=1$

$$a_3 = \frac{-a_0}{(r+3)^2-n^2} = 0$$

iii) $m=2$

$$a_4 = \frac{-a_2}{[(r+4)^2-n^2]} = \frac{a_0}{[(r+2)^2-n^2]\,[(r+4)^2-n^2]} \quad \text{---- so on.}$$

The solⁿ is

$$y = x^n\,[a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots\cdots]$$

$$y = x^m a_0 \left[1 - \frac{1}{(r+2)^2-n^2}x^2 - \frac{x^4}{[(r+2)^2-n^2][(r+4)^2-n^2]} \cdots\right]$$

for $r = n$

$$y = x^n a_0 \left[1 - \frac{x^2}{4(n+1)} + \frac{x^4}{4^2\,2!\,(n+1)(n+2)} \cdots\cdots\right]$$

for $r = -n$

$$y = a_0 x^{-n} \left[1 - \frac{1}{4(-n+1)}x^2 + \frac{x^4}{4^2\,2!\,(-n+1)(-n+2)} \cdots\cdots\right]$$

is the required solⁿ.

Bessel's fun^n; $J_n(x)$ of First kind

The sol^n of Bessel's Equation $x^2y'' + xy' + (x^2-n^2)y = 0$ is

$$y = a_0 x^n \left[1 - \frac{x^2}{2\cdot 2(n+1)} + \frac{x^4}{2\cdot 4\cdot 2^2 (n+1)(n+2)} - \cdots \right.$$

$$\left. \cdots + (-1)^r \frac{x^{2r}}{(2^r r!)\, 2^r (n+1)(n+2)\cdots(n+r)} \right]$$

$$= a_0 x^n \sum_{r=0}^{\infty} \frac{(-1)^r x^{2r}}{2^{2r} r!\, (n+1)(n+2)\cdots(n+r)}$$

Thus

a_0 arbitary Constant

$$a_0 = \frac{1}{2^n\, n!}$$

$J_n(x) =$

$$\frac{1}{2^n \Gamma(n+1)} \sum_{r=0}^{\infty} \frac{(-1)^r x^{n+2r}}{2^{2r} r!\, (n+1)(n+2)\cdots(n+r)}$$

$[\Gamma(n+1) = n!]$

$= \left(\frac{x}{2}\right)^n$

$$J_n(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\, \Gamma(n+r+1)} \left(\frac{x}{2}\right)^{2r+n}$$

$$\boxed{J_n(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\, (n+r)!} \left(\frac{x}{2}\right)^{2r+n}}$$

for $n = 0$ & 1, $J_0(x)$, $J_1(x)$ are given as

$$J_0(x) = 1 - \frac{x^2}{2^2} + \frac{x^4}{2^2 4^2} - \frac{x^6}{2^2 4^2 6^2} - \cdots$$

$$J_1(x) = \frac{x}{2} - \frac{x^3}{2^2 3^2} + \frac{x^5}{2^2 4^2 6^2} - \cdots$$

we have

$$J_{-n}(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{r! \, \Gamma(-n+r+1)} \left(\frac{x}{2}\right)^{2r-n}$$

$J_{-n}(x)$ called as Bessel's funⁿ of second kind.

→ when n not an integer then $J_n(x)$ and $J_{-n}(x)$ are two independent solⁿ and general solⁿ is given by

$$y = c_1 J_n(x) + c_2 J_{-n}(x).$$

Show that the Bessel funⁿ $J_n(x)$ is an even funⁿ if n is even otherwise odd.

we have

$$J_n(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{r! \, \Gamma(n+r+1)} \left(\frac{x}{2}\right)^{n+2r}$$

replace x by $(-x)$

$$J_{n}(-x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{r! \, \Gamma(n+r+1)} \left(\frac{-x}{2}\right)^{n+2r}$$

Case I If n is even

then $n+2r$ also even $\left(-\frac{x}{2}\right)^{n+2r} = \left(\frac{x}{2}\right)^{n+2r}$

$J_n(-x) = J_n(x)$ i.e. even funⁿ.

Case II if n is odd

then $n+2r$ is odd $\left(-\frac{x}{2}\right)^{n+2r} = -\left(\frac{x}{2}\right)^{n+2r}$

$J_n(-x) = -J_n(x)$

i.e. odd funⁿ

Prove that $\lim_{x\to 0} \frac{J_n(x)}{x^n} = \frac{1}{2^n \Gamma n+1} = \frac{1}{2^n n!}$

Sol^n. we have

$$J_n(x) = \frac{x^n}{2^n \Gamma n+1}\left[1 - \frac{x^2}{2(2n+2)} + \frac{x^4}{2.4(2n+2)(2n+4)} - \cdots\right]$$

on taking lim both sides when $x \to 0$

$$\lim_{x\to 0} \frac{J_n(x)}{x^n} = \lim_{x\to 0} \frac{1}{2^n \Gamma n+1}\left[1 - \frac{x^2}{2(2n+2)} + \frac{x^4}{2.4(2n+2)(2n+4)}\right]$$

$$\boxed{\lim_{x\to 0} \frac{J_n(x)}{x^n} = \frac{1}{2^n \Gamma n+1}}$$

Prove that $J_{-n}(x) = (-1)^n J_n(x)$

we have

$$J_{-n}(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\, \Gamma r-n+1} \left(\frac{x}{2}\right)^{2r-n}$$

$$= \underbrace{\sum_{r=0}^{r=n-1} \frac{(-1)^r}{r!\, \Gamma r-n+1} \left(\frac{x}{2}\right)^{2r-n}}_{0} + \sum_{r=n}^{\infty} \frac{(-1)^r \left(\frac{x}{2}\right)^{2r-n}}{r!\, \Gamma r-n+1}$$

$$\sum_{r=n}^{\infty} \frac{(-1)^r \left(\frac{x}{2}\right)^{2r-n}}{r!\, \Gamma r-n+1}$$

Put $r = n+k$

$$= \sum_{k=0}^{\infty} \frac{(-1)^{n+k} \left(\frac{x}{2}\right)^{n+2k}}{(n+k)!\, \Gamma k+1}$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^n (-1)^k \left(\frac{x}{2}\right)^{n+2k}}{k!\, \Gamma n+k+1}$$

$$= (-1)^n J_n(x)$$

Prove that

i) $J_{1/2}(x) = \sqrt{\frac{2}{\pi x}} \sin x$

ii) $J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}} \cos x$

$\rightarrow$ $n = 1/2$

$$J_{1/2} = \frac{x^{1/2}}{2^{1/2}\,\Gamma\left(\frac{1}{2}+1\right)}\left[1 - \frac{x^2}{2\cdot 2\left(\frac{1}{2}+1\right)} + \frac{x^4}{2\cdot 4\cdot 2^2\left(\frac{1}{2}+1\right)\left(\frac{1}{2}+2\right)} - - - -\right]$$

$$\frac{\sqrt{x}}{\sqrt{2}\,\Gamma 3/2}\left[1 - \frac{x^2}{2\cdot 3} + \frac{x^4}{2\cdot 3\cdot 4\cdot 5} - - - -\right]$$

$$\frac{\sqrt{x}}{\sqrt{2}\,\frac{1}{2}\,\Gamma\frac{1}{2}}\cdot\frac{1}{x}\left[x - \frac{x^3}{3!} + \frac{x^5}{5!} - - - - - -\right]$$

$$\frac{\sqrt{2}}{\sqrt{x}\,\sqrt{\pi}} \sin x \Rightarrow \sqrt{\frac{2}{x\pi}} \sin x.$$

$\rightarrow$ $n = -1/2$

$$J_{-1/2} = \frac{x^{-1/2}}{2^{-1/2}\,\Gamma\left(\frac{-1}{2}+1\right)}\left[1 - \frac{x^2}{2\cdot 2\left(-\frac{1}{2}+1\right)} + \frac{x^4}{2\cdot 4\cdot 2^2\left(\frac{-1}{2}+1\right)\left(-\frac{1}{2}+2\right)} - - - - - - -\right]$$

$$\sqrt{\frac{2}{x\pi}}\left[1 - \frac{x^2}{2!} + \frac{x^4}{4!} - - - -\right]$$

$$\sqrt{\frac{2}{\pi x}} \cos x$$

Proved

Generating funⁿ for $J_n(x)$

Acc. to this the coffⁿ of t^n in the Expansion of $e^{\frac{x}{2}(t-\frac{1}{t})}$ is $J_n(x)$.

we have
$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} - - - - - -$$

$$e^{\frac{xt}{2}} = 1 + \left(\frac{xt}{2}\right) + \frac{1}{2!}\left(\frac{xt}{2}\right)^2 + \frac{1}{3!}\left(\frac{xt}{2}\right)^3 - - - - \quad \text{(I)}$$

$$e^{-\frac{x}{2t}} = 1 + \left(\frac{-x}{2t}\right) + \frac{1}{2!}\left(\frac{-x}{2t}\right)^2 + \frac{1}{3!}\left(\frac{-x}{2t}\right)^3 - - - - \quad \text{(II)}$$

On multiplying eqⁿ (I) & (II), we get

$$e^{\frac{x}{2}(t-\frac{1}{t})} = \left[1 + \left(\frac{xt}{2}\right) + \frac{1}{2!}\left(\frac{xt}{2}\right)^2 + - - -\right] \times \left[1 - \left(\frac{x}{2t}\right) + \frac{1}{2!}\left(\frac{x}{2t}\right)^2 - -\right]$$

Now the coffⁿ of t^n in this, we get

$$\frac{1}{n!}\left(\frac{x}{2}\right)^n \left[1 - \frac{x^2}{2!\,(n+1)\,2^2} + \frac{x^4}{2!\,(n+2)!\,2^4} - - - -\right]$$

$$= J_n(x)$$

Thus,

$$e^{\frac{x}{2}(t-\frac{1}{t})} = J_0(x) + tJ_1(x) + t^2 J_2(x) + - - - - - -$$

$$\boxed{e^{\frac{x}{2}(t-\frac{1}{t})} = \sum_{n=-\infty}^{\infty} t^n J_n(x)}$$

This called as the Generating funⁿ

Recurrence Formulae

① $x J'_n = n J_n - x J_{n+1}$

we have

$$J_n(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,\Gamma(n+r+1)} \left(\frac{x}{2}\right)^{n+2r}$$

Diffⁿ w.r.t x we get

$$J'_n(x) = \sum \frac{(-1)^r (n+2r)}{r!\,\Gamma(n+r+1)} \left(\frac{x}{2}\right)^{n+2r-1} \left(\frac{1}{2}\right) \cancel{\sum \frac{(-1)^r\, 2r}{2\cdot r!\,\Gamma(n+r+1)}} \left(\frac{x}{2}\right)^{n+2r-1}$$

$$x J'_n(x) = n \sum \frac{(-1)^r}{r!\,\Gamma(n+r+1)} \left(\frac{x}{2}\right)^{n+2r} + x \sum \frac{(-1)^r\, 2r}{2\cdot r!\,\Gamma(n+r+1)} \left(\frac{x}{2}\right)^{n+2r-1}$$

$$= n J_n(x) + x \sum_{r=1}^{\infty} \frac{(-1)^r}{2r\,(r-1)!\,\Gamma(n+r+1)} \left(\frac{x}{2}\right)^{n+2r-1}$$

$$= n J_n(x) + x \sum_{t=0}^{\infty} \frac{(-1)^{t+1}}{t!\,\Gamma(n+t+2)} \left(\frac{x}{2}\right)^{n+2t+1} \quad [\text{Let } r-1=t$$

$$= n J_n(x) - x \sum_{t=0}^{\infty} \frac{(-1)^t}{t!\,\Gamma((n+1)+t+1)} \left(\frac{x}{2}\right)^{(n+1)+2t}$$

$$\boxed{x J'_n(x) = n J_n(x) - x J_{n+1}(x)}$$

Proved

② $x J'_n(x) = -n J_n(x) + x J_{n-1}(x)$

we have $J_n(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,(n+r)!} \left(\frac{x}{2}\right)^{n+2r}$

diffⁿ w.r.t x, we get

$$J'_n(x) = \sum_{r=0}^{\infty} \frac{(-1)^r (n+2r)}{r!\,(n+r)!} \times \frac{1}{2} \left(\frac{x}{2}\right)^{n+2r-1}$$

$$= \sum_{r=0}^{\infty} \frac{(-1)^r [(2n+2r) - n]}{r!\,(n+r)!} \left(\frac{x}{2}\right)^{n+2r}$$

$$= \sum_{r=0}^{\infty} \frac{(-1)^r \, 2(n+r)}{r!\,(n+r)(n+r-1)!} \left(\frac{x}{2}\right)^{n+2r} - n \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,(n+r)!} \left(\frac{x}{2}\right)^{n+2r}$$

$$= x \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,\Gamma(n-1)+r+1} \left(\frac{x}{2}\right)^{(n-1+2r)} - n J_n(x)$$

$$\boxed{x J_n'(x) = x J_{n-1}(x) - n J_n(x)}$$

Proved

(3) $2 J_n' = J_{n-1} - J_{n+1}$

Adding Relation (1) & (2), we get

$$2x J_n'(x) = x J_{n-1}(x) - x J_{n+1}$$

$$\boxed{2 J_n'(x) = J_{n-1}(x) - J_{n+1}}$$

Proved

(4) $2n J_n(x) = x (J_{n-1}(x) + J_{n+1})$

Subtract (2) from (1)

$$\boxed{2n J_n(x) = x [J_{n-1}(x) + J_{n+1})}$$

Proved

(5) $\frac{d}{dx}\left(x^{-n} J_n(x)\right) = -x^{-n} J_{n+1}$

we have $x J_n'(x) = n J_n(x) - x J_{n+1}(x)$

Multiply by x^{-n-1}

$$x^{-n} J_n'(x) = n x^{-n-1} J_n(x) - x^{-n} J_{n+1}(x)$$

$$\boxed{\frac{d}{dx}\left[x^{-n} J_n(x)\right] = -x^{-n} J_{n+1}}$$

Hermite funⁿ

Eqⁿ of the form

$$\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2ny = 0$$

called the Hermite diffⁿ Eqⁿ.

Let the series solⁿ be

$$y = \sum_{m=0}^{\infty} a_m x^{m+r}$$

$$y' = (m+r) \sum_{m=0}^{\infty} a_m x^{m+r-1}$$

$$y'' = (m+r)(m+r-1) \sum_{m=0}^{\infty} a_m x^{m+r-2}$$

we get

$$(m+r)(m+r-1) \sum_{m=0}^{\infty} a_m x^{m+r-2} + \left[-2(m+r) + 2n\right] \sum_{m=0}^{\infty} a_m x^{m+r} = 0$$

$$(m+r)(m+r-1) \sum_{m=0}^{\infty} a_m x^{m+r-2} - 2(m+r-n) \sum_{m=0}^{\infty} a_m x^{m+r} = 0$$

1) coffⁿ of x^{r-2}

$$r(r-1)a_0 = 0 \qquad a_0 \neq 0$$

$$\boxed{r = 0, 1}$$

1) coffⁿ of x^{m+r}

$$(m+r+2)(m+r+1)\, a_{m+2} - 2(m+r-n)\, a_m = 0$$

$$a_{m+2} = \frac{2(m+r-n)\, a_m}{(m+r+1)(m+r+2)}$$

Case II $r = 0$

$$a_{m+2} = \frac{2(m-n)}{(m+1)(m+2)} a_m$$

i) $m = 0$

$$a_2 = \frac{-2n}{2} a_0 = -n a_0$$

ii) $m = 1$

$$a_3 = \frac{2(1-n)}{2\cdot 3} a_1 = \frac{-2(n-1)}{3!} a_1$$

iii) $m = 2$

$$a_4 = \frac{2 \cdot 2(2-n)}{2\cdot 3\cdot 4} = \frac{(2)^2 n(n-2)}{4!} a_0$$

$\rightarrow$ when $a_1 = 0$

$$a_3 = a_5 = a_7 - - - - = 0$$

$\rightarrow$ when $a_1 \neq 0$

$$y = \sum_{m=0}^{\infty} x^m a_m$$

$$y = \left[a_0 + a_1 x + a_2 x^2 + a_3 x^3 - - - - - - \right.$$

$$a_0 \left[1 - \frac{2n}{2!} x^2 + \frac{2^2 n(n-2)}{4!} x^4 - - - - - - \right]$$

$$+ a_1 x \left[1 - \frac{2(n-1)}{3!} x^2 + \frac{2^2 (n-1)(n-3)}{5!} - - - \right]$$

$\longrightarrow$ (B)

Case II $\quad r = 1$

$$a_{m+2} = \frac{2(m+1-n)}{(m+3)(m+2)} a_m$$

$\left\{ \begin{array}{l} a_1 m(m+1) \\ \quad \downarrow \quad \Downarrow \neq 0 \\ =0 \\ \text{when } m=1 \end{array} \right.$

i) $m = 0$

$$a_2 = \frac{2(1-n)}{1\cdot2\cdot3} a_0 = -\frac{2(n-1)}{3!} a_0$$

ii) $m = 1$

$$a_3 = \frac{2(2-n)\, a_1}{} = 0$$

iii) $m = 2$

$$a_4 = \frac{-2(n-3)}{4\cdot5} a_2 = \frac{+2^2(n-1)(n-3)}{5!} a_0$$

Hence the solⁿ.

$$y = a_0 x \left[1 - \frac{2(n-1)}{3!} x^2 + \frac{2^2(n-1)(n-3)}{5!} x^4 - \cdots \cdots \right] \quad \text{(A)}$$

Eqⁿ (A) is contained in Eqⁿ (B)

Thus, the complete solⁿ is given by

$$y = A\left[1 - \frac{2n}{2!}x^2 + \frac{2^2 n(n-2)}{4!}x^4 + \cdots\right] + B\left[1 - \frac{2(n-1)}{3!}x^2 + \frac{2^2(n-1)(n-3)}{5!}x^4 \cdots\right]$$

The solⁿ of this Eqⁿ is called as the Hermite polynomial.

Laguerres funⁿ

The Eqⁿ

$$x\frac{d^2y}{dx^2} + (1-x)\frac{dy}{dx} + ny = 0$$

called as Laguerres diffⁿ Eqⁿ.

Let $y = \sum_{m=0}^{\infty} a_m x^{m+r}$

$y' = (m+r)\sum_{m=0}^{\infty} a_m x^{m+r-1}$

$y'' = (m+r)(m+r-1)\sum_{m=0}^{\infty} a_m x^{m+r-2}$

$$\left[(m+r)(m+r-1) + (m+r)\right]\sum_{m=0}^{\infty} a_m x^{m+r-1} - \left[\cancel{x}(m+r) \cancel{+} n\right]\sum_{m=0}^{\infty} a_m x^{m+r} = 0$$

$$(m+r)^2 \sum_{m=0}^{\infty} a_m x^{m+r-1} - (m+r \cancel{+} n)\sum_{m=0}^{\infty} a_m x^{m+r} = 0$$

i) coffⁿ of x^{r-1}

$r^2 a_0 = 0$

$\boxed{r = 0, 0}$

ii) coffⁿ of x^{m+r}

$(m+r+1)^2 a_{m+1} - (m+r \cancel{+} n) a_m = 0$

$$a_{m+1} = \frac{(m+r \cancel{+} n)}{(m+r+1)^2} a_m$$

for $r=0$

$$a_{m+1} = \frac{(m-n)}{(m+1)^2} a_m$$

$$a_2 = \frac{(1-n)}{2^2} a_1 = +\frac{n(n-1)}{(2!)^2} a_0$$

$$a_1 = \frac{-n}{1!} a_0$$

$$a_3 = -\frac{(n-2)}{3^2} a_2 = -\frac{n(n-1)(n-2)}{(3!)^2} a_0$$

we have

$$y = \sum_{m=0}^{\infty} a_m x^m$$

$$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$$

$$= a_0 \left[1 - nx + \frac{n(n-1)}{(2!)^2} x^2 - \frac{n(n-1)(n-2)}{(3!)^2} x^3 - \cdots \right]$$

$$= a_0 \sum_{m=0}^{\infty} \frac{(-1)^m \, x^m}{(m!)^2 (n-m)!}$$

if $a_0 = n!$ then sol^n y called Laguerre Poly^M

$$L_n(x) = (-1)^n \left[x^n - \frac{n^2 x^{n-1}}{1!} + \frac{n^2 (n-1)^2 x^{n-2}}{2!} + \cdots \right]$$

Gen form

1. $(1-x^2)y'' - 2xy' + n(n+1)y = 0$ Legendre
2. $x^2y'' + xy' + (x^2 - n^2)y = 0$ Bessel
3. $y'' - 2xy' + 2ny = 0$ Hermite
4. $xy'' + (1-x)y' + ny = 0$ Laguerre